

COVARIANT FIELD THEORY FOR SELF-DUAL STRINGS

N. Berkovits¹

*Inst. de Física Teórica, Univ. Estadual Paulista
Rua Pamplona 145, São Paulo, SP 01405-900, Brasil*

W. Siegel²

*Institute for Theoretical Physics
State University of New York, Stony Brook, NY 11794-3840*

ABSTRACT

We give a gauge and manifestly $SO(2,2)$ covariant formulation of the field theory of the self-dual string. The string fields are gauge connections that turn the super-Virasoro generators into covariant derivatives.

¹ Internet address: nberkovi@ift.unesp.br.

² Internet address: siegel@insti.physics.sunysb.edu.

1. Introduction

There are three known kinds of string theories: Those with critical (uncompactified) dimension: (1) $D=26$ (“ $N=0$ ”), which has various fundamental problems (divergences, tachyons, no fermions, etc.); (2) $D=10$ (“ $N=1$ ”), which is now thought to be a misleading formulation of a $D=11$ theory that includes supermembranes; and (3) $D=4$ [1] (“ $N=2$ ” [2]), which describes self-dual massless theories in 2 space and 2 time dimensions [3,4]. (Note that critical/uncompactified dimension characterizes a string theory since, by embedding one string into another, the number of worldsheet supersymmetries can be altered. It is not clear if the results in this paper will be useful for $N=2$ strings which come from embeddings of the $D=26$ and $D=10$ strings.)

The last type of string (the topic of this paper), because of its 2 time dimensions, has lent itself to various interpretations. Clearly unitarity cannot be applied in the usual way, and usually is ignored. 4D Lorentz invariance is not manifest in the usual $N=2$ formulation, and therefore also has largely been neglected, even though the self-dual field theories with which it is identified have a Lorentz covariant definition. Directly related to the loss of manifest Lorentz invariance is the loss of gauge invariance: The $N=2$ formulations correspond to certain light-cone gauges, which are not always the best choice for analyzing such theories, particularly for such non-perturbative solutions as instantons. Since spin is ignored, statistics is also ignored; besides, unitarity and Lorentz invariance are the usual justifications for their relation. The $N=2$ string is also equivalent [5] to the $N=4$ string [6], but although the latter formulation is manifestly Lorentz covariant, its complicated ghost structure has not been completely worked out, and therefore its existence is seldom recognized. Even dimensional analysis is a problem [7] since, e.g., pure self-dual Yang-Mills contains no dimensionful coupling constant, unlike the field theory action used in [3,4].

The precise definition of this string theory depends strongly on the motivation for its consideration. Up to now, all the work on the noncovariant formulation of the self-dual string has been associated with the fact that it implies classical field equations for self-dual Yang-Mills theory or self-dual gravity [3,4] (but not self-dual gravity coupled to self-dual Yang-Mills [4]) in light-cone gauges. These equations of motion can be used to derive the classical equations of motion of wide classes of integrable models in lower dimensions, as well as study certain properties of solutions of the classical equations in four dimensions. However, most of the known 4D solutions (multi-instantons) require more general gauges to be written explicitly [8]. (For example, explicit n -Eguchi-Hanson and n -Taub-NUT solutions to self-dual gravity would

require solving $2n$ -th-order polynomial equations in this coordinate system, and thus can be explicit only for $n=1,2$ [9].) Furthermore, the identification as the self-dual part of some non-self-dual theory at the quantum level requires dimensional analysis to be consistent (e.g., for the renormalization group). In particular, the fact that these self-dual field theories can be interpreted as (Wick rotations of) truncations of the corresponding non-self-dual theories [10,11] implies that this string theory actually can be used to help understand physical theories in $3+1$ dimensions, and perform perturbative and nonperturbative calculations in them.

For this purpose, it is useful to find a method of applying 2D conformal field theory that preserves Lorentz and gauge invariances. Traditionally, the string field or wave function in any string theory has been assumed to be a scalar (or at least a one-component field), with all excitations described by its dependence on its arguments. However, this is not a physical requirement of the theory, but an assumption of the conformal field theory description. The same assumption is generally not made for the quantum mechanics or quantum field theory of particles, and it is not clear that such a requirement would aid in the evaluation of Feynman diagrams. Since the purpose of two-dimensional conformal field theory in string theory is perturbation, one might consider calculational rules for string S-matrices that allow for “indices” in addition to the obvious coordinates. These indices are analogous to the Chan-Paton factors which appear in open string theory and which have no conformal field theory justification. (Although it is true that Chan-Paton factors can be associated with fermionic coordinates living on the worldsheet boundary, their influence is always calculated by simply multiplying the group-theory factors into the amplitude, rather than by calculating worldline propagators for the fermions, etc.) Just as the $SO(32)$ Chan-Paton factors of the light-cone-gauge open superstring are required for $SO(9,1)$ Lorentz invariance, the indices in the self-dual string are needed for $SO(2,2)$ Lorentz-invariance. In earlier papers such indices were associated with $N=2$ theories to restore Lorentz invariance (and also allow supersymmetry) [10,7,11]. One immediate improvement over the no-index formulation, even at the classical level and in the usual light-cone gauges, is that the equations of motion can consistently describe self-dual gravity coupled to self-dual Yang-Mills theory (see section 2).

The purpose of this paper is to associate Lorentz indices with string fields (or wave functions) in such a way as to give a string description of self-dual theories while preserving gauge invariance and manifest Lorentz invariance. The usual string descriptions of these theories are related to light-cone gauge choices (with the associated elimination of auxiliary string fields). The new formulation of the string theory,

and its relation to the conventional ones, is closely analogous to the known treatment of the particle field theory describing just the massless fields. We therefore use, as our guide for covariantizing the string theory, the covariant description of the particle theory, which we review in the following section. As for the particle field theory, two different light-cone gauges are possible for the string field theory, corresponding to the polynomial (cubic vertex) and nonpolynomial (Wess-Zumino-like) formulations. The string field theory has already been formulated in the latter gauge [12], so we review it in section 3. It is the formulation to which we apply the covariantization, as described in section 4. In the final section we discuss supersymmetry and ghosts.

2. Self-dual Yang-Mills theory

For purposes of perturbation theory, we can describe ordinary Yang-Mills theory as a perturbation about self-dual Yang-Mills theory [13] (see [11] for a light-cone approach): We can write the Yang-Mills Lagrangian in first-order form as [14]

$$\mathcal{L} = G^{\alpha\beta} F_{\alpha\beta} + g^2 G^2$$

where g is the usual Yang-Mills coupling, $G^{\alpha\beta} = G^{\beta\alpha}$ is an anti-self-dual tensor, and

$$F_{\alpha\beta} \equiv [\nabla_{\alpha}^{\dot{\gamma}}, \nabla_{\beta\dot{\gamma}}] = \partial_{(\alpha}^{\dot{\gamma}} A_{\beta)\dot{\gamma}} + [A_{\alpha}^{\dot{\gamma}}, A_{\beta\dot{\gamma}}]$$

is the anti-self-dual-part of the usual Yang-Mills field strength F . Here $\alpha = \pm$ is an $\text{SL}(2, \mathbb{C})$ Weyl spinor index, and $\dot{\alpha} = \pm$ is its complex conjugate, in 3+1 dimensions. To make the action real, we can Wick rotate to 2+2 dimensions, where α becomes an $\text{SL}(2)$ ($\text{SL}(2, \mathbb{R})$) index, and $\dot{\alpha}$ an $\text{SL}(2)'$ index. ($\text{SO}(3,1) = \text{SL}(2, \mathbb{C})$, $\text{SO}(2,2) = \text{SL}(2) \otimes \text{SL}(2)$.) Spinor indices are raised and lowered with the $\text{SL}(2)$ metric $C_{\alpha\beta}$ (antisymmetric and Hermitian). Note that although $\text{SL}(2) = \text{SU}(1,1)$, the $\text{SU}(1,1)$ notation common in the $\text{N}=2$ string theory literature assumes a particular representation where the $\text{U}(1,1)$ metric is diagonal. This differs from the Majorana representation natural to $\text{SL}(2)$ notation, where the $\text{U}(1,1)$ metric is antisymmetric. In particular, in the diagonal representation a spinor χ^{α} satisfies $(\chi^{\pm})^* = \chi^{\mp}$, while in the antisymmetric (Majorana) representation $(\chi^{\pm})^* = \chi^{\pm}$. In this paper we'll generally refrain from using complex conjugation explicitly, since this makes Wick rotation easier, so our notation can easily be specialized to any representation.

Elimination of G by its equation of motion produces the usual Lagrangian, up to a total derivative. (The action is then real in either 3+1 or 2+2.) On the other hand, we can keep G , and treat \hbar ($\mathcal{L} \rightarrow \mathcal{L}/\hbar$) and g^2 as independent expansion parameters.

To lowest order in g^2 (i.e., $g = 0$), we have a theory that describes self-dual Yang-Mills theory, in the sense that G is then a Lagrange multiplier that enforces self-duality of F . However, G itself is propagating, as required by Lorentz invariance: Propagating helicity $+1$ in the self-dual part of F requires propagating helicity -1 multiplying it in the action. This perturbation expansion in g^2 is natural in the sense that the simplest tree and one-loop amplitudes in Yang-Mills theory are those where (almost) all the external helicities are the same, and the amplitudes become progressively more complicated as more helicities change sign. Similar remarks apply to self-dual gravity, where the non-self-dual Lagrangian can be written in differential-form notation as [14]

$$\mathcal{L} = e^{\alpha\dot{\gamma}} \wedge e^{\beta}_{\dot{\gamma}} \wedge (d\omega_{\alpha\beta} + \kappa^2 \omega_{\alpha}^{\delta} \wedge \omega_{\delta\beta})$$

where $e^{\alpha\dot{\beta}} = dx^m e_m^{\alpha\dot{\beta}}$ is the vierbein form (the analog of A above) and $\omega^{\alpha\beta} = dx^m \omega_m^{\alpha\beta}$ is the anti-self-dual part of the Lorentz connection form (the analog of G above).

In fact, almost all the amplitudes at $g = 0$ or $\kappa = 0$ vanish, so this term in the action is very similar to a kinetic term: In the self-dual theories described by the above actions, (1) all the tree amplitudes vanish on shell except for the three-point (but it also vanishes on-shell in 3+1 dimensions for kinematic reasons), and (2) all amplitudes vanish at more than one loop, since \hbar counting arguments show that any L -loop diagram must have $1-L$ external Lagrange multiplier lines. (The classical action, which goes as $1/\hbar$, is linear in Lagrange multipliers, so in the effective action the order in Lagrange multipliers is minus the order in \hbar .) Furthermore, in any of the supersymmetric versions of the self-dual theories (N=1 supersymmetry or greater), all the one-loop amplitudes also vanish.

From now on we restrict ourselves to the self-dual theories ($g = \kappa = 0$, or lowest order in that perturbation expansion). There are two light-cone gauges for analyzing the self-duality condition $F_{\alpha\beta} = 0$ in Yang-Mills theory (see [11] for an analysis at the quantum level): (1) a gauge proposed by Yang [15], which gives field equations resembling a 2D Wess-Zumino model, and (2) a gauge that gives a quadratic field equation, found by Leznov, Mukhtarov, and Parkes [16]. In both cases, we first choose the light-cone gauge and then solve the $F_{++} = 0$ part of the self-duality condition:

$$\left. \begin{array}{l} \text{gauge } A_{++}^{\dot{\cdot}} = 0 \\ F_{++} = 0 \Rightarrow A_{+-}^{\dot{\cdot}} = 0 \end{array} \right\} \Rightarrow A_{+\dot{\alpha}} = 0$$

The two cases differ in which of the remaining two equations is solved as a constraint, and which is left as a field equation: In the Yang case,

$$\begin{aligned} F_{--} = 0 & \Rightarrow A_{-\dot{\alpha}} = e^{-\phi} \partial_{-\dot{\alpha}} e^{\phi} \\ F_{+-} = 0 & \Rightarrow \partial_+^{\dot{\alpha}} (e^{-\phi} \partial_{-\dot{\alpha}} e^{\phi}) = 0 \end{aligned}$$

while in the LMP case

$$\begin{aligned} F_{+-} = 0 & \Rightarrow A_{-\dot{\alpha}} = \partial_{+\dot{\alpha}} \phi \\ F_{--} = 0 & \Rightarrow -i\Box\phi + (\partial_+^{\dot{\alpha}} \phi)(\partial_{+\dot{\alpha}} \phi) = 0 \end{aligned}$$

Note that in the LMP case the light-cone “time” derivatives $\partial_{-\dot{\alpha}}$ appear only in the kinetic term $\Box\phi$, while the Yang case is more like a Wess-Zumino model, with such derivatives included in the interaction term.

If we denote the surviving component of $G^{\alpha\beta}$ by $\tilde{\phi}$ (G^{+-} in the Yang case, G^{--} in the LMP case), then the Lagrangian becomes just $\tilde{\phi}$ times the ϕ field equation. Note that G (and thus $\tilde{\phi}$) has engineering dimension 2, while ϕ is dimensionless. Also, the Lorentz transformations of ϕ and $\tilde{\phi}$ differ. (This is especially clear in the LMP case, where they even have different weights under the unbroken $GL(1)$ subgroup of the $SL(2)$ acting on the undotted spinor indices.) Thus, it is not possible to write an action in terms of just ϕ that reproduces the above field equations, without violating Lorentz invariance and introducing a dimensionful coupling. (Even with $\tilde{\phi}$, Lorentz invariance is not manifest: The Lorentz transformations are nonlinear in the fields.) Furthermore, using a single field ϕ destroys the correspondence with ordinary Yang-Mills theory, as described by a perturbation about the self-dual theory. Perturbatively, the (off-shell) tree graphs agree, since they are the classical field equations. The only difference is the labeling of the external lines: Calling one external line $\tilde{\phi}$ and the rest ϕ gives the same Feynman diagram as labeling all lines ϕ (although the interpretation is different). However, the 1-loop graphs of the single-field theory differ by a factor of 1/2, and it has nonvanishing higher-loop graphs that have no apparent relation to Yang-Mills theory. As for open superstrings with different gauge groups, the only difference in the theories is the index structure (in this case, 1-valued index vs. 2-valued), but this makes all the difference in the quantum corrections.

This analysis has been extended to self-dual gravity, and to self-dual gravity coupled to Yang-Mills theory (as well as their supersymmetric versions) [7]. In the case of gravity, the analog of the Yang gauge is the Plebański gauge [17], which in this case gives quadratic field equations. While the analog of the LMP gauge [7] again

gives quadratic field equations, they differ from the Plebański gauge by the absence of “—” derivatives in the interaction term. (In N=2 string theory, the Plebański gauge arises if world-sheet instantons are ignored, while the LMP-like gauge follows if they are included [18].) In both cases, the gravitational 3-point vertex is the square of the corresponding Yang-Mills one (4 derivatives instead of 2).

In general, then, the differences between the various actions are rather small, at least at the level of the propagator and 3-point vertices. (Higher-point vertices exist only for Yang-Mills theory, and only in the Yang gauge, whether with or without a Lagrange multiplier.) Explicitly, the kinetic term in the lagrangian is always of the form

$$\mathcal{L}_2 \sim \phi_1 \square \phi_2$$

while the cubic term representing Yang-Mills coupling is

$$\mathcal{L}_3 \sim \phi_1 (\partial \phi_2) (\partial \phi_3)$$

and that for gravitational coupling is

$$\mathcal{L}'_3 \sim \phi_1 (\partial \partial \phi_2) (\partial \partial \phi_3)$$

The explicit indices on the derivatives differ for Yang/Plebanski gauges vs. LMP gauges; here we will instead focus on the difference with or without Lagrange multipliers. Without Lagrange multipliers: (1) The kinetic term has ϕ_1 and ϕ_2 the same in any such term, independent of helicity (whether graviton, gluon, or their superpartners); (2) the Yang-Mills coupling \mathcal{L}_3 also has all fields the same, namely 3 gluons (or a supersymmetric generalization); and (3) the gravitational coupling \mathcal{L}'_3 has either 3 gravitons, or 2 gluons and 1 graviton (or supersymmetric generalization). On the other hand, when Lagrange multipliers are introduced, each of the 3 kinds of terms is linear in them (and thus either linear or quadratic in the usual fields). In that case, it's simpler to describe the couplings in terms of helicity: +2 for (self-dual) graviton, -2 for its Lagrange multiplier, +1 for photon, etc. Then \mathcal{L}_2 has fields with helicity summing to 0, \mathcal{L}_3 has any fields with helicity summing to +1, and \mathcal{L}'_3 has any summing to +2. (This applies also to the supersymmetric cases.)

There are several levels of Lorentz invariance a description of a theory can have: (1) The highest is when the theory is described in terms of an action that is manifestly Lorentz invariant. (2) The next level, as results for example when a noncovariant gauge is chosen or some auxiliary fields are noncovariantly eliminated, is when there is an action that is still invariant, but for which the Lorentz transformations are

nonlinear (and perhaps even nonlocal). This is the case for the light-cone gauge actions described above when the Lagrange multiplier fields are included. (3) An even lower level is that for the corresponding case when the Lagrange multipliers are absent; the action is then not Lorentz invariant in any sense, but the field equations are Lorentz covariant in the sense of the previous level. (4) The lowest level lacks any kind of Lorentz invariance for even the field equations. This is the case for the coupling of the closed and open N=2 strings, as found in [4], which we now discuss in more detail. There the term “self-dual” is loosely applied, since the Plebański equation gets a source term from the gluons, and thus no longer describes self-dual gravity. The resulting field equation has no Lorentz covariant analog. The vertices are exactly those described in the previous paragraph. On the other hand, the Lorentz covariant action with Lagrange multipliers that we have discussed reproduces these vertices, except for the different index structure. The necessity for the Lagrange multipliers for a covariant interpretation is clear from the covariant actions given above: For self-dual gravity coupled to self-dual Yang-Mills, the actions given above (gravitationally covariantized for the Yang-Mills terms) give the field equations

$$S = eed\omega + GF \quad \Rightarrow \quad 0 = \begin{cases} \delta S/\delta\omega &= & dee \\ \delta S/\delta G &= & F \\ \delta S/\delta A &= & \nabla G \\ \delta S/\delta e &= & ed\omega + GF \end{cases}$$

where all indices are implicit. Thus, the self-duality equations for the vierbein and Yang-Mills are unaffected (except for covariantization of the latter), while the self-dual Yang-Mills energy-momentum tensor $G_{\alpha\beta}F_{\alpha\beta}^{\cdot\cdot}$ appears in the field equation for ω . This clearly corresponds to the index structure described for the light-cone Lagrangian terms described in the previous paragraph, where the fields e , A , G , ω have helicities $+2$, $+1$, -1 , -2 . Thus, a simple relabeling of fields has strong implications even at the level of classical field equations.

3. Noncovariant version of self-dual string field theory

In a paper by one of the authors [19], it was shown how to construct an open string field theory action for any critical N=2 superconformal representation. This action differs from the standard open string field theory action [20], $\Phi Q\Phi + \lambda\Phi^3$, in that it is built directly out of N=2 matter fields and does not require worldsheet ghosts. This is possible since, after twisting, N=2 ghosts carry no central charge and decouple from scattering amplitudes. This ghost-free description of N=2 strings

was developed by one of the authors with C. Vafa [12] and is extremely useful for calculating N=2 scattering amplitudes [12,21].

In the ghost-free description of N=2 strings, it is useful to note that any critical N=2 representation contains generators of a “small” N=4 superconformal algebra. For the self-dual representation of the N=2 string, the left-moving N=4 generators are:

$$\begin{aligned} T &= (\partial_z X^{\dot{\alpha}\dot{\beta}})(\partial_z X_{\dot{\alpha}\dot{\beta}}) + \psi^{\dot{\alpha}\ddot{\beta}} \partial_z \psi_{\dot{\alpha}\ddot{\beta}} \\ G^{\alpha\ddot{\beta}} &= \psi^{\dot{\gamma}\ddot{\beta}} \partial_z X_{\dot{\gamma}}^{\alpha} \\ J^{\ddot{\alpha}\ddot{\beta}} &= \psi^{\dot{\gamma}\ddot{\alpha}} \psi_{\dot{\gamma}}^{\ddot{\beta}} \end{aligned}$$

where $X^{\dot{\alpha}\dot{\beta}}$ and $\psi^{\dot{\alpha}\ddot{\beta}}$ are the usual string coordinates. α (SL(2)) and $\dot{\alpha}$ (SL(2)') are the usual 4D Weyl spinor indices, while $\ddot{\alpha}$ (the SL(2)'' of $J^{\ddot{\alpha}\ddot{\beta}}$) is the world-sheet internal index. The $c = 6$ N=2 generators are T , $G^{+\ddot{-}}$, $G^{-\ddot{+}}$, $J^{+\ddot{-}}$. After twisting $T \rightarrow T' = T + \frac{1}{2}J^{+\ddot{-}}$, T , $G^{\alpha\ddot{-}}$ and $J^{+\ddot{-}}$ carry spin two, $G^{\alpha\ddot{+}}$ and $J^{+\ddot{+}}$ carry spin one, and $J^{+\ddot{+}}$ carries spin zero. Furthermore, $\psi^{\dot{\alpha}\ddot{+}}$ is now spin zero while $\psi^{\dot{\alpha}\ddot{-}}$ is spin one.

In this formulation, only SL(2)' is completely preserved manifestly. Although in this paper we work in an N=2 formulation, we'll find that both spacetime SL(2)'s can be preserved in the string field theory after adding indices on the string field. However, SL(2)'' remains broken to the usual local U(1) (or GL(1)) symmetry of the worldsheet, generated by $J^{+\ddot{-}}$. (The $\ddot{\alpha} = \pm$ indices refer to the U(1) charge.)

As was described in [12], these generators can be used to compute N -point scattering amplitudes on surfaces of genus (field-theory-loops) L and instanton number n_I where $|n_I| \leq 2L - 2 + N$. The most relevant scattering amplitude for open string field theory is the three-point tree amplitude at zero instanton number, which is given by:

$$\langle \Phi(z_1)(Q^+ \Phi(z_2))(Q^- \Phi(z_3)) \rangle \quad (3.1)$$

where $Q^\alpha \Phi$ signifies the contour integral of spin-one $G^{\alpha\ddot{+}}$ around the vertex operator Φ and $\langle \rangle$ signifies the two-dimensional correlation function on a sphere. Note that this correlation function vanishes unless the two zero-modes of $\psi^{\dot{\alpha}\ddot{+}}$ are present.

To be a physical vertex operator, $\Phi(X, \psi)$ must be U(1)-neutral (in the N=2 topological method, worldsheet U(1) charge is equal to spacetime ghost number) and satisfy the linearized equation of motion $Q^\alpha Q_\alpha \Phi = 0$. (This implies that Φ is a weight-zero N=2 primary field.) Unlike the usual vertex operator in string theory, Φ is defined to be bosonic. Note that the contour integral of Q^+ anticommutes with the

contour integral of Q^- , so (3.1) is invariant under the linearized gauge transformation $\delta\Phi = Q^\alpha\Lambda_\alpha$. As in all open string theories, Φ carries Chan-Paton factors which will be suppressed throughout this paper.

Up to gauge transformations, the only momentum-dependent U(1)-neutral vertex operator satisfying $Q^\alpha Q_\alpha\Phi = 0$ is $\Phi = \exp(ik^{\alpha\dot{\alpha}}X_{\alpha\dot{\alpha}})$ where $k^{\alpha\dot{\alpha}}k_{\alpha\dot{\alpha}} = 0$. After performing the correlation functions over the N=2 matter fields (remembering that $\psi^{\dot{\alpha}+}$ has a zero-mode), one finds that (3.1) produces the usual three-point tree amplitude $k_2^{+\dot{\alpha}}k_3^{-\dot{\alpha}}f^{I_1I_2I_3}$ where $f^{I_1I_2I_3}$ is the structure constant for the Chan-Paton factors and k_r is the momentum of the r -th state.

To construct a string field theory action, it is natural to generalize the on-shell vertex operator to an off-shell string field Φ which is an arbitrary function of $X(\sigma)$, $\psi(\sigma)$. Note that the U(1) (GL(1)) charge of Φ is related to the ghost number of the spacetime field. If one chooses the Majorana representation for SL(2)'' spinors (implying that our choice of $J^{+\dots}$ for the N=2 super-Virasoro algebra corresponds to a GL(1) subgroup of this SL(2), rather than a U(1) subgroup), the reality condition on the string field is the usual one:

$$\Phi(\sigma)^\dagger = \Phi(\pi - \sigma) \quad (3.2)$$

Note that in this representation, twisting $T \rightarrow T' = T + \frac{1}{2}J^{+\dots}$ commutes with hermitian conjugation. One can also “Wick-rotate” this choice to the diagonal representation for (the U(1,1) metric of) the SL(2)'' spinors (implying that $J^{+\dots}$ corresponds to a U(1) subgroup), but then hermitian conjugation must be accompanied by an SL(2) transformation to restore the original twist [19].

For the string field theory action to be correct, the quadratic term in the action should enforce the linearized equation of motion $Q^\alpha Q_\alpha\Phi = 0$, while the cubic term should produce the correct on-shell three-point amplitude. Finally, the action should contain a gauge invariance whose linearized form is $\delta\Phi = Q^\alpha\Lambda_\alpha$.

The quadratic and cubic terms in the action are easily found to be of the form

$$\frac{1}{2} \int \left[\frac{1}{2}i(Q^\alpha\Phi)(Q_\alpha\Phi) - \frac{1}{3}\Phi\{Q^+\Phi, Q^-\Phi\} \right]$$

where “ \int ” means integration over all the modes of X and ψ . However, there is no nonlinear version of the gauge transformation $\delta\Phi = Q^\alpha\Lambda_\alpha$ which leaves this action invariant. Knowing the equations of motion for self-dual Yang-Mills in Yang gauge, an obvious guess for the nonlinear generalization of $Q^\alpha Q_\alpha\Phi = 0$ is $Q^-(e^{-\Phi}Q^+e^\Phi) = 0$,

where multiplication of string fields is always performed using Witten's half-string overlap. If ϕ is the component of Φ which depends only on the zero-mode of X , then $Q^-(e^{-\Phi}Q^+e^{\Phi})$ contains the term $\psi^{\dot{\alpha}\ddot{+}}\partial_{\dot{\alpha}}^-(e^{-\phi}\psi^{\dot{\beta}\ddot{+}}\partial_{\dot{\beta}}^+e^{\phi}) = \psi^{\dot{\alpha}\ddot{+}}\psi_{\dot{\alpha}}^{\dot{+}}\partial_{\dot{\beta}}^{-\dot{\beta}}(e^{-\phi}\partial_{\dot{\beta}}^+e^{\phi})$, which is $\psi^{\dot{\alpha}\ddot{+}}\psi_{\dot{\alpha}}^{\dot{+}}$ times the self-dual equation of motion in Yang gauge.

The action which produces this equation of motion is a straightforward generalization of the Wess-Zumino model [22] where the two-dimensional derivatives ∂_z and $\bar{\partial}_{\bar{z}}$ are replaced by Q^+ and Q^- . The string field theory action is

$$\frac{1}{2} \int \left[(e^{-\Phi}Q^+e^{\Phi})(e^{-\Phi}Q^-e^{\Phi}) - \int_0^1 dt (e^{-t\Phi}\partial_t e^{t\Phi}) \{e^{-t\Phi}Q^+e^{t\Phi}, e^{-t\Phi}Q^-e^{t\Phi}\} \right] \quad (3.3)$$

In addition to producing the correct linearized equations of motion and three-point tree amplitude, this action contains the nonlinear gauge invariance,

$$\delta e^{\Phi} = (Q^+\Lambda_+)e^{\Phi} + e^{\Phi}(Q^-\Lambda_-) \quad (3.4)$$

which generalizes the linearized gauge invariance $\delta\Phi = Q^{\alpha}\Lambda_{\alpha}$.

4. Lorentz covariance

In this section, we show how to “covariantize” the field theory action for the self-dual representation of the N=2 string. It is still unclear if the covariantization procedure will be useful for other N=2 superconformal representations. The results of the previous section for the N=2 open string are clearly analogous to those for self-dual Yang-Mills in the Yang gauge, with the identification

$$Q_{\alpha} \leftrightarrow \partial_{\dot{\alpha}\dot{\beta}}$$

I.e., in the string field theory the dotted spinor indices are dropped, the antisymmetry of the $SL(2)'$ metric on those indices being replaced with the anticommutativity of the “BRST operators” Q_{α} . (The dotted spinor indices reappear if one expands out the $\psi^{\dot{\alpha}\ddot{+}}$ dependence.)

This suggests that to recover 4D Lorentz invariance, one needs to place a two-valued index on the string field:

$$\Phi_1 = \tilde{\Phi}, \quad \Phi_2 = \Phi$$

where $\tilde{\Phi}$ plays the role of the Lagrange multiplier and Φ plays the role of the self-dual field. Furthermore, the string field action in Yang gauge needs to be modified to

$\int \tilde{\Phi} Q^-(e^{-\Phi} Q^+ e^{\Phi})$. Note that except for the index structure and numerical (permutation) factors, this action has the same quadratic and cubic terms as (3.3), and the same linearized gauge transformation (see below). The change in the index structure has the effect of multiplying the usual conformal field theory calculation by a factor $\delta_{1-n,L} 2^L$ where n is the number of tilded vertex operators and L is the number of loops.

We can extend the analogy to the manifestly Lorentz covariant formulation by proposing the new string field theory action

$$S = \int G^{\alpha\beta} F_{\alpha\beta}, \quad F_{\alpha\beta} = \{\nabla_\alpha, \nabla_\beta\}, \quad \nabla_\alpha = Q_\alpha + A_\alpha$$

which is invariant under the gauge transformations

$$\nabla'_\alpha = e^K \nabla_\alpha e^{-K}, \quad G'_{\alpha\beta} = e^K G_{\alpha\beta} e^{-K} + \nabla^\gamma \Omega_{\gamma\alpha\beta}$$

where K is arbitrary and $\Omega_{\gamma\alpha\beta}$ is symmetric in its indices. A_α and $G^{\alpha\beta}$ are our new string fields, now carrying $SL(2)$ spinor indices as well as implicit Yang-Mills gauge-group indices (and with the same string coordinates as arguments). Note that the on-shell (surviving) field strength $F_{\alpha\beta}$ has no simple string field analog. This is standard in string field theory: E.g., in $N=0$ open string field theory, the field strength that vanishes on-shell ($Q\Phi + \Phi * \Phi$) is obvious, while the nonvanishing string field strength has no local expression.

The string field theory action of the previous section can now be rederived from this covariant action by the same methods described in section 2. However, since the BRST operators Q_α have nontrivial kernels (in contrast to the partial derivatives $\partial_{\alpha\beta}$), new gauge invariances arise upon solving the constraints, in close analogy to 4D $N=1$ super Yang-Mills theory [19]. (See [23] for a review.) In this case, the gauge invariances arise because of the unphysical “massive” fields, absent in the discussion of section 2.

Explicitly, we find

$$F_{++} = 0 \quad \Rightarrow \quad A_+ = 0$$

in an appropriate gauge. (I.e., $F_{++} = 0$ implies A_+ is pure gauge.) However, this does not completely fix the gauge: We are left with the residual gauge invariance

$$0 = \delta A_+ = -Q_+ K \quad \Leftarrow \quad K = Q_+ \Xi$$

In particular, this applies for the gauge transformation of G .

For the next step, in the Yang gauge,

$$F_{--} = 0 \quad \Rightarrow \quad A_- = e^{-\Phi} Q_- e^{\Phi}$$

This introduces the gauge invariance

$$(e^{\Phi})' = e^{\Lambda} e^{\Phi}, \quad Q_- \Lambda = 0 \quad \Leftarrow \quad \Lambda = Q_- \Theta$$

The complete gauge transformations for Φ are now

$$(e^{\Phi})' = e^{\Lambda} e^{\Phi} e^{-K}; \quad K = Q_+ \Xi, \quad \Lambda = Q_- \Theta$$

Finally, the field equation is

$$0 = F_{+-} = Q_+(e^{-\Phi} Q_- e^{\Phi})$$

The Lagrangian then reduces to $\tilde{\Phi} F_{+-}$, where $\tilde{\Phi} = G_{+-}$. Applying these results, the gauge transformation for $\tilde{\Phi}$ is then

$$\delta \tilde{\Phi} = [Q_+ \Xi, \tilde{\Phi}] + Q_+ \Omega_{-+-} + Q_- \Omega_{++-} + \{e^{-\Phi} Q_- e^{\Phi}, \Omega_{++-}\}$$

So, as claimed above, the linearized gauge transformations of $\int \tilde{\Phi} F_{+-}$ are of the form $\delta \Phi = Q^{\alpha} \Lambda_{\alpha}$ and $\delta \tilde{\Phi} = Q^{\alpha} \tilde{\Lambda}_{\alpha}$.

On the other hand, we can also find a new light-cone string action by going to an LMP gauge:

$$F_{+-} = 0 \quad \Rightarrow \quad A_- = Q_+ \Phi$$

This introduces the *Abelian* gauge invariance

$$\delta \Phi = \Lambda, \quad \Lambda = Q_+ \Theta$$

The complete gauge transformation for Φ is now

$$\delta \Phi = (Q_- \Xi + [Q_+ \Xi, \Phi]) + Q_+ \Theta$$

The field equation is now polynomial:

$$0 = F_{--} = -i Q^{\alpha} Q_{\alpha} \Phi + 2(Q_+ \Phi)^2$$

and the Lagrangian is $\tilde{\Phi} F_{--}$, $\tilde{\Phi} = G_{++}$. The gauge transformation of $\tilde{\Phi}$ is

$$\delta \tilde{\Phi} = [Q_+ \Xi, \tilde{\Phi}] + Q_+ \Omega_{-++} + Q_- \Omega_{+++} + \{Q_+ \Phi, \Omega_{+++}\}$$

5. Supersymmetry and ghosts

The supersymmetric generalizations of self-dual theories have also been analyzed [10,7]. Self-dual Yang-Mills theory can be treated as a truncation of self-dual N=4 super Yang-Mills theory, where A and G are in the same supermultiplet. (So can self-dual super Yang-Mills theories for N<4.) The component action in 2+2 dimensions (where all fields are real) is

$$\mathcal{L} = \frac{1}{2}G^{\alpha\beta}F_{\alpha\beta} + \xi^{i\alpha}\nabla_{\alpha}^{\dot{\beta}}\chi_{i\dot{\beta}} + \epsilon^{ijkl}(\frac{1}{8}\phi_{ij}\square\phi_{kl} + \frac{1}{4}\phi_{ij}\chi_k^{\dot{\alpha}}\chi_{l\dot{\alpha}})$$

where $i, j, k, l = 1, \dots, 4$ are the internal SL(4) indices of N=4 supersymmetry (not to be confused with the Yang-Mills group indices, which are still implicit), and ϕ_{ij} is antisymmetric. Furthermore, self-dual Yang-Mills theory, self-dual gravity, and self-dual gravity coupled to self-dual Yang-Mills theory all can be treated as truncations of gauged self-dual N=8 supergravity (with Yang-Mills gauge group SO(8)). In light-cone gauges, the vertices (and, of course, the propagators) are identical to the nonsupersymmetric cases: Spin appears effectively as an internal symmetry index.

This helicity-independence of the couplings also has an explanation in terms of N=2 strings: Spectral flow is usually interpreted as allowing the identification of states with different boundary conditions (which would normally be associated with different spins, and thus different statistics). However, these states can be distinguished if they are assigned different helicities. (The assignment of helicities is somewhat arbitrary in N=2 string theory; in fact, the usual continuous helicity representations of the Poincaré group [24] can be associated with these self-dual theories, but only if one abandons the possibility of manifest Lorentz covariance.) So, rather than using spectral flow to say there's only one state, it can be interpreted as a stronger version of supersymmetry that implies helicity-independence of the couplings.

As for the nonsupersymmetric case, the above component action can be straightforwardly generalized to string field theory by dropping dotted spinor indices and replacing $\partial \rightarrow Q$:

$$\mathcal{L} = \frac{1}{2}G^{\alpha\beta}F_{\alpha\beta} + \xi^{i\alpha}\nabla_{\alpha}\chi_i + \epsilon^{ijkl}(\frac{1}{8}\phi_{ij}\square\phi_{kl} + \frac{1}{4}\phi_{ij}\chi_k\chi_l)$$

where now $\square \equiv \frac{1}{2}\nabla^{\alpha}\nabla_{\alpha}$. The component fields describing helicities +1, +1/2, 0, -1/2, -1 are now A_{α} , χ_i , ϕ_{ij} , $\xi^{i\alpha}$, $G^{\alpha\beta}$. Half of the supersymmetry transformations (those not involving $F_{\alpha\beta}$ explicitly) can also be generalized:

$$\delta A_{\alpha} = \epsilon^i{}_{\alpha}\chi_i$$

$$\begin{aligned}
\delta\chi_i &= \epsilon^{j\alpha}\nabla_\alpha\phi_{ji} \\
\delta\phi_{ij} &= -\epsilon_{ijkl}\epsilon^{k\alpha}\xi^l{}_\alpha \\
\delta\xi^i{}_\alpha &= \frac{1}{2}\epsilon^{i\beta}G_{\alpha\beta} + \frac{1}{2}\epsilon^m{}_\alpha\epsilon^{ijkl}[\phi_{jk}, \phi_{lm}] \\
\delta G_{\alpha\beta} &= -\epsilon^i{}_{(\alpha}[\xi^j{}_{\beta)}, \phi_{ij}]
\end{aligned}$$

There is an interesting analogy between supersymmetry and the Zinn-Justin-Batalin-Vilkovisky formalism: The minimal field theory Lagrangian for the nonsupersymmetric self-dual string, including antifields and the ghosts for just the Yang-Mills gauge symmetry, is

$$\mathcal{L} = G^{\alpha\beta}F_{\alpha\beta} + A^{*\alpha}\nabla_\alpha C + C^*C^2$$

This is very similar to the supersymmetric action (without antifields), with the identification

$$\chi_i \leftrightarrow C, \quad \phi_{ij} \leftrightarrow C^*, \quad \xi^{i\alpha} \leftrightarrow A^{*\alpha}$$

This suggests a generalization of the $GL(N)$ internal symmetry of N -extended supersymmetry to $SL(N|1)$. (For $N < 4$, the Lagrange multipliers form a separate supermultiplet from the other fields, and the ϵ_{ijkl} 's can be absorbed.) Because of the nontrivial cohomology of Q_α there is an infinite number of generations of ghosts; the $SL(N|1)$ symmetry might simplify their classification.

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